Dynamic Analysis of Rotor Blade System

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***Abstract*—To make the rotation of the rotor blade more accurate, the elastic deformation in the system is considered in this paper. Firstly, the number and direction of the generalized coordinates are determined by the freedom degree of the blade and the finite element method and Lagrange equation are used to build the dynamical equation of the rotor blade system. Then, we obtain the position vector, the mass matrix, the derivative of Mass Matrix on time and other coefficient matrix according to the principle of flexible multibody dynamics. Finally, the violation correction method is used to get the numerical solution of the dynamical equation in the simulation. Additionally, the angular displacement of the hub around the main axis, the angular displacement of the blade end and elastic deformation in the y-direction and z-direction are analyzed to prove the correctness of the model.**

***Keyword*—rotor blade, elastic deformation, finite element, flexible multibody dynamics.**

I. INTRODUCTION

THE study of flexible multibody dynamics began in the late 1950s during which the US launched the first artificial satellite. In the past 20 years, due to its very close relationship with the engineering field, such as aerospace, aviation, machinery, vehicles and robots, even with the sports, flexible multibody dynamics have caused great concern and become an active study field of theoretical and applied mechanics. Flexible multibody dynamics researches the dynamics behavior of the systems consisted of deformable objects and rigid body during wide-range spatial movement. Flexible multibody dynamics focus on "flexible", and study the interaction between the objects' deformation and overall rigidity movement. The character that deformation movement and rigidity movement occur and couple simultaneously is the core feature of flexible multibody dynamics.

A novel mathematical formulation capable of treating the problems of maneuvering and control of flexible multibody structures is developed [1]. The authors focused on an approach to the study of the dynamics and control of large flexible space structures which comprised of subassemblies, a subject of considerable contemporary interest [2]. A flexible multibody dynamics formulation to analyze the vibration of hard disk drives is presented [3-4]. The inverse-dynamic model of mobile multibody systems articulated with joints and wheels is built and an easily implementable algorithm which is based on Newton–Euler recursive dynamics is proposed [5]. This authors presented several iterative algorithms regarding the dynamic analysis of multibody systems [6].

However, there is few literatures investigating the dynamic of the rotor blade using the flexible multibody dynamics. Moreover, the research on helicopter fall behind that on Fixed-wing aircraft, the reason is mainly that the awareness of the helicopter rotor aerodynamic is not sufficient. Based on this, the dynamical equation of the rotor blade system is built using the finite element method and the Lagrange equation. Additionally, the mass matrix and other coefficient matrix are obtained. After that, the simulation is used to verify the correctness of the dynamical model.

II. SYSTEM MODEL

Helicopter rotor is composed of hub and several blades. The hub mounted on the rotor shaft, while the blade which likes the slender wing connects to the hub. The hub and blade rotated together with the helicopter engine rotation, and the interaction between the blade and the surrounding air can create the thrust along the rotor shaft. Additionally, the blade contains flap, lag and torsion, and all these motions coupled together to formed a complex structure-dynamics problem.

The helicopter rotor not only rotates around the engine shaft, but also has Elastic deformations due to its flexible. According to the motion characteristics of the rotor blade in space, we can determine the number of generalized coordinates, and build the finite element model of the rotor blade in Fig. 1.

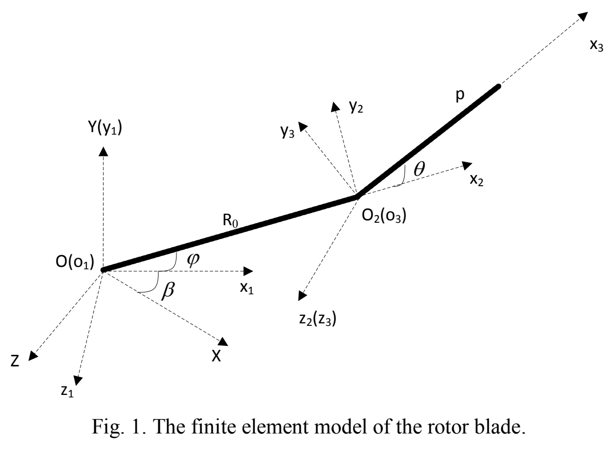

Fig. 1. The finite element model of the rotor blade.

In Fig. 1, OXYZ is the inertial coordinate frame, while $o_1x_1y_1z_1$, $o_2x_2y_2z_2$ and $o_3x_3y_3z_3$ are the moving coordinates. The $o_1x_1y_1z_1$ frame rotates around OY axis with vector R_0,

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and the rotation angle is β; the $o_2x_2y_2z_2$ frame rotates around o1z1 axis with vector R_0 , and the rotation angle is β; the $o_3x_3y_3z_3$ frame whose origin coincides with that of the $o_1x_1y_1z_1$ frame rotates around o_2z_2 axis with vector R_0 , and the rotation angle is θ.

III. DYNAMICAL EQUATION AND COEFFICIENT MATRIX OF THE ROTOR BLADE

A. Dynamical Equation

The dynamical equation of flexible multibody systems is given as follows:

$$M\ddot{q}+\dot{M}\dot{q}-\frac{1}{2}\dot{q}^T\frac{\partial M}{\partial q}\dot{q}+Kq=Q_F \tag{1}$$

In helicopter rotor system, the blade is hinge connected to the hub, and flap around the hub. The hub rotates around the engine shaft. Due to the damping effect in these motions, the damping coefficient matrix and the constraint equation are introduced, and dynamical equation of the helicopter rotor blade can be written as follows:

$$M\ddot{q}+D\dot{q}+Kq+C_q^T\lambda=Q_F+Q_V \tag{2}$$

where M is the mass matrix, Q_F is the generalized external force vector, q is the generalized coordinate vector, $\dot{q}$ is the generalized speed vector, D is the damping coefficient matrix, K is the stiffness coefficient matrix, C_q is the constraint Jacobian matrix, λ is the Lagrange multipliers vector, and $C_q^T\lambda$ is the generalized constraint force.

Submitting $Q_V=-\dot{M}\dot{q}+\frac{1}{2}\dot{q}^T(\frac{\partial M}{\partial q})\dot{q}$ into Equation (2), the dynamical equation can be rewritten as:

$$M\ddot{q}+(D+\dot{M})\dot{q}+Kq+C_q^T\lambda=Q_F+\frac{1}{2}\dot{q}^T(\frac{\partial M}{\partial q})\dot{q} \tag{3}$$

Where $\dot{M}$ is the partial derivative of mass matrix on time, $\frac{\partial M}{\partial q}$ is the derivative of mass matrix on generalized coordinate.

B. Position Vector

The displacement of the arbitrary point on beam element in the inertial coordinate frame can be given as

$$R_p=R_0+A_1A_2A_3(u_0+Nq_f) \tag{4}$$

where $R_p=[x_0 \quad y_0 \quad z_0]$ is the position at the start time, $A_1=\begin{bmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \cos\beta\end{bmatrix}$ is the rotation matrix from $o_1x_1y_1z$ to OXYZ, $A_2=\begin{bmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}$ is the rotation matrix from $o_2x_2y_2z_2$ to OXYZ, $A_3=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$ is the rotation matrix from $o_3x_3y_3z_3$ to OXYZ, β is the rotation angle around OY axis, φ is the rotation angle around oz_1 axis, θ is the rotation angle around o_2z_2 axis, u is the vector of point p after distortion in moving coordinates $o_3x_3y_3z_3$, u_0 is the vector of point p without distortion in moving coordinates $o_3x_3y_3z_3$, and is the position vector of point p by distortion in moving coordinates $o_3x_3y_3z_3$.

The flight conditions of helicopter is different with different β, φ and θ. The state of analysis in this paper is as shown in formula (4). At this time, the helicopter fly forward, backward, or fly by hover. When $\beta=0$, the helicopter is in a static state; when β is a fixed value and $\varphi=0$, the helicopter is in the air hover state; when β and φ are both constant, the helicopter in the air flight state.

According to the finite element principle [7-8], the displacement of any point on the element can be represented using the interpolation method. That is, the displacements of the element nodes are used to represent the elastic deformation at any point on the element.

Setting u_0 as the position vector before beam deformation, N as the shape function of beam element, and q_f as displacement vector of beam element, the elastic deformation at any point on the element can be given as follows:

$$N_f=Nq_f \tag{5}$$

where

$$N_f=\begin{bmatrix}n_1 & 0 & 0 & 0 & 0 & 0 & n_2 & 0 & 0 & 0 & 0 & 0\\ 0 & n_3 & 0 & 0 & 0 & n_4 & 0 & n_5 & 0 & 0 & 0 & n_6\\ 0 & 0 & n_3 & 0 & n_7 & 0 & 0 & 0 & n_5 & 0 & n_8 & 0\end{bmatrix}$$

is the shape function of beam element, and $n_1=1-\frac{x}{l}$, $n_2=\frac{x}{l}$, $n_3=1-\frac{3x^2}{l^2}+\frac{2x^3}{l^3}$, $n_4=x-\frac{2x^2}{l}+\frac{x^3}{l^2}$, $n_5=\frac{3x^2}{l^2}-\frac{2x^3}{l^3}$, $n_6=-\frac{3x^2}{l}+\frac{x^3}{l^2}$, $n_7=-n_4$, $n_8=-n_6$, x is the abscissa of any point in moving coordinates $o_3x_3y_3z_3$, l is the length of the beam element.

Space beam element has two nodes (i node and j node), each node has six degrees of freedom along the three coordinate axes and three axes around the rotation, thus the node displacement vector can be expressed as

$$q_f=\begin{bmatrix}q_{i1} & q_{i2} & q_{i3} & q_{i4} & q_{i5} & q_{i6} & q_{j1} & q_{j1} & q_{j1} & q_{j1} & q_{j1} & q_{j1}\end{bmatrix}^T$$

C. Mass Matrix

Calculating the derivative on time in equation (4), we can get the speed of point p as follows:

$$\begin{aligned}\dot{R}_p&=\dot{R}_0+A_{1\beta}\dot{\beta}A_2A_3U+A_1A_{2\varphi}\dot{\varphi}A_3U\\&+A_1A_2A_{3\theta}\dot{\theta}U+A_1A_2A_3N\dot{q}_f\end{aligned} \tag{6}$$

Due to the constant u_0, we can get $\dot{u}_0=0$ and $\dot{u}=N\dot{q}_f$. Base on this, the equation (6) can be reorganized as:

$$\dot{R}_p = [I \;\; A_{1\beta}A_2A_3U \;\; A_1A_{2\varphi}A_3U \;\; A_1A_2A_{3\theta}U \;\; A_1A_2A_3N]\begin{bmatrix}\dot{R}_0\\ \dot{\beta}\\ \dot{\varphi}\\ \dot{\theta}\\ \dot{q}_f\end{bmatrix} \tag{7}$$

where $\dot{R}_p$ is the velocity vector at any point on the beam, $\dot{q}_f$ is the derivative of beam element's displacement on time, $A_{3\theta} = \begin{bmatrix}-\sin\theta & -\cos\theta & 0\\ \cos\theta & -\sin\theta & 0\\ 0 & 0 & 0\end{bmatrix}$ is the derivative of A_3 on θ, $A_{1\beta} = \begin{bmatrix}-\sin\beta & 0 & \cos\beta\\ 0 & 0 & 0\\ -\cos\beta & 0 & -\sin\beta\end{bmatrix}$ is the derivative of A1 on β , $A_{2\beta} = \begin{bmatrix}-\sin\varphi & -\cos\varphi & 0\\ \cos\varphi & -\sin\varphi & 0\\ 0 & 0 & 0\end{bmatrix}$ is the derivative of A2 on φ , $\dot{q} = \begin{bmatrix}\dot{R}_0 & \dot{\beta} & \dot{\varphi} & \dot{\theta} & \dot{q}_f\end{bmatrix}^T$ is the generalized velocity vector.

The kinetic energy of the flexible body is given as:

$$T = \frac{1}{2}\int_V \rho \dot{R}_p^{\,T} \dot{R}_p \, dV = \frac{1}{2}\dot{q}^T M \dot{q} \tag{8}$$

where ρ and V are the density and volume of the beam element respectively.

Submitting Equation (7) into Equation (8), we can kinetic energy can be rewritten as:

$$\begin{aligned} T &= \frac{1}{2}\int_V \rho \dot{R}_p^{\,T} \dot{R}_p \, dV \\ &= \frac{1}{2}\int_V \dot{q}^T \begin{bmatrix} I \\ A_{1\beta}A_2A_3U \\ A_1A_{2\varphi}A_3U \\ A_1A_2A_{3\theta}U \\ A_1A_2A_3N \end{bmatrix} [I \;\; A_{1\beta}A_2A_3U \;\; A_1A_{2\varphi}A_3U \;\; A_1A_2A_{3\theta}U \;\; A_1A_2A_3N]\dot{q}\, dV \\ &= \frac{1}{2}\dot{q}^T M \dot{q} \end{aligned} \tag{9}$$

From Equation (8), the mass matrix of the rotor blade M can be obtained.

D. The derivative of Mass Matrix on time

The derivative of mass matrix on time is given as follows:

$$\dot{M} = \frac{\partial M}{\partial q}\frac{\partial q}{\partial t} = \frac{\partial M}{\partial q}\dot{q} \tag{10}$$

Since β, θ, and φ all contains one generalized coordinate, q_f has 12 generalized coordinates, and R_0 has three generalized coordinates, $\dot{q}$ can be written as

$$\dot{q} = \begin{bmatrix}\dot{X}_0 & \dot{Y}_0 & \dot{Z}_0 & \dot{\beta} & \dot{\varphi} & \dot{\theta} & \dot{q}_{i1} & \dot{q}_{i2} & \dot{q}_{i3} \\ \dot{q}_{i4} & \dot{q}_{i5} & \dot{q}_{i6} & \dot{q}_{j1} & \dot{q}_{j2} & \dot{q}_{j3} & \dot{q}_{j4} & \dot{q}_{j5} & \dot{q}_{j6}\end{bmatrix} \tag{11}$$

The generalized coordinate is vector, thus the above equation is the partial derivative of the mass matrix on each generalized coordinate. Based on this, the derivative of mass matrix on time can be rewritten as

$$\dot{M} = \sum_{i=1}^{18}\frac{\partial M}{\partial q_i}\dot{q}_i = \begin{bmatrix}\dot{m}_{11} & \dot{m}_{12} & \dot{m}_{13} & \dot{m}_{14} & \dot{m}_{15}\\ & \dot{m}_{22} & \dot{m}_{23} & \dot{m}_{24} & \dot{m}_{25}\\ & & \dot{m}_{33} & \dot{m}_{34} & \dot{m}_{35}\\ & & & \dot{m}_{44} & \dot{m}_{45}\\ & & & & \dot{m}_{55}\end{bmatrix} \tag{12}$$

where

$$\frac{\partial M}{\partial q_i} = \begin{bmatrix}\frac{\partial m_{11}}{\partial q_i} & \frac{\partial m_{12}}{\partial q_i} & \frac{\partial m_{13}}{\partial q_i} & \frac{\partial m_{14}}{\partial q_i} & \frac{\partial m_{15}}{\partial q_i}\\ & \frac{\partial m_{22}}{\partial q_i} & \frac{\partial m_{23}}{\partial q_i} & \frac{\partial m_{24}}{\partial q_i} & \frac{\partial m_{25}}{\partial q_i}\\ & & \frac{\partial m_{33}}{\partial q_i} & \frac{\partial m_{34}}{\partial q_i} & \frac{\partial m_{35}}{\partial q_i}\\ & & & \frac{\partial m_{44}}{\partial q_i} & \frac{\partial m_{45}}{\partial q_i}\\ & & & & \frac{\partial m_{55}}{\partial q_i}\end{bmatrix},$$

$$\dot{m}_{11} = \sum_{i=1}^{18}\frac{\partial m_{11}}{\partial q_i}\dot{q}_i ,$$

$$\dot{m}_{12} = \dot{m}_{21} = \sum_{i=1}^{18}\frac{\partial m_{12}}{\partial q_i}\dot{q}_i, \; \dot{m}_{13} = \dot{m}_{31} = \sum_{i=1}^{18}\frac{\partial m_{13}}{\partial q_i}\dot{q}_i,$$

$$\dot{m}_{14} = \dot{m}_{41} = \sum_{i=1}^{18}\frac{\partial m_{14}}{\partial q_i}\dot{q}_i, \; \dot{m}_{15} = \dot{m}_{51} = \sum_{i=1}^{18}\frac{\partial m_{15}}{\partial q_i}\dot{q}_i,$$

$$\dot{m}_{22} = \sum_{i=1}^{18}\frac{\partial m_{22}}{\partial q_i}\dot{q}_i, \; \dot{m}_{23} = \dot{m}_{32} = \sum_{i=1}^{18}\frac{\partial m_{23}}{\partial q_i}\dot{q}_i,$$

$$\dot{m}_{24} = \dot{m}_{42} = \sum_{i=1}^{18}\frac{\partial m_{24}}{\partial q_i}\dot{q}_i, \; \dot{m}_{25} = \dot{m}_{52} = \sum_{i=1}^{18}\frac{\partial m_{25}}{\partial q_i}\dot{q}_i,$$

$$\dot{m}_{33} = \sum_{i=1}^{18}\frac{\partial m_{33}}{\partial q_i}\dot{q}_i, \; \dot{m}_{34} = \sum_{i=1}^{18}\frac{\partial m_{34}}{\partial q_i}\dot{q}_i, \; \dot{m}_{35} = \sum_{i=1}^{18}\frac{\partial m_{35}}{\partial q_i}\dot{q}_i,$$

$$\dot{m}_{44} = \sum_{i=1}^{18}\frac{\partial m_{44}}{\partial q_i}\dot{q}_i, \; \dot{m}_{45} = \sum_{i=1}^{18}\frac{\partial m_{45}}{\partial q_i}\dot{q}_i, \; \dot{m}_{55} = \sum_{i=1}^{18}\frac{\partial m_{55}}{\partial q_i}\dot{q}_i .$$

E. Partial Derivatives of Kinetic Energy on Generalized Coordinates

Based on the expression of kinetic energy in equation (9), the partial derivative of kinetic energy on the generalized coordinate can be expressed as

$$\frac{\partial T}{\partial q} = \frac{1}{2}\dot{q}^T\frac{\partial M}{\partial q}\dot{q} = \begin{bmatrix}\frac{1}{2}\dot{q}^T\frac{\partial M}{\partial q_1}\dot{q}\\ \cdots\cdots\\ \frac{1}{2}\dot{q}^T\frac{\partial M}{\partial q_{18}}\dot{q}\end{bmatrix} \tag{13}$$

F. Constraint Equation and Jacobian Matrix

The Lagrange multipliers method is used to build the constraint equation in this paper. The general form of the constraint equation is $C(q,t)=0$. The distance between the blade root and the rotation center is constant, thus the constraint equation of the rotation center is given as:

$$C(q,t) = R_0 + A_1 A_2 A_3 u_f^i - l_0 \tag{14}$$

where R_0 is the position vector of the rotation center, u_f^i is the elastic deformation, l_0 is the distance between the blade root and the rotation center.

According the relationship of the location change between each object, the constraint equation with variational form can be obtained as $C_q \delta q = 0$. Then, the constraint Jacobian matrix can be derived as：

$$C_q = \begin{bmatrix} c_{11} & \bullet & \bullet & c_{1n} \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ c_{m1} & \bullet & \bullet & c_{mn} \end{bmatrix} \tag{15}$$

where $c_{ij} = \dfrac{\partial c_i}{\partial q_i}$, m is the number of the constraint equations, n is the number of generalized coordinate. In multibody system, the element C_q is the the partial derivative of the i-th constraint equation on the j-th generalized coordinate.

G. Stiffness Matrix

The virtual work of the cell in the object can be written as:

$$\delta W = -\int_V \sigma^T \delta\varepsilon dV \tag{16}$$

where σ and ε are the stress and strain. For the linear isotropic materials, we can get $\sigma = E_T \varepsilon$, where E_T is the elastic quantity matrix. The strain can be got by $\varepsilon = Du_f = DNq_f$, where D is the partial differential operators of the strain-displacement.

Submitting the expression ofεinto equation (17), the virtual work can be rewritten as:

$$\delta W = -q_f^T K_{ff} \delta q_f \tag{17}$$

From the equation (17), the beam element stiffness matrix can be derived as:

$$K_{ff} = \int (DN)^T EDNdV \tag{18}$$

H. Damping Matrix

The aerodynamics model is two-dimensional steady model in this paper. The introduction of damping coefficient has two functions. One is to balance the torque of the helicopter engineer to make the rotor shaft uniform rotation. The other is supporting a big attenuation coefficient to the flap of the blade, increase the iterative rate of the flap and reduce the computation time.

The damping coefficients of the rotor shaft and flapping ream are $D_{\theta\theta}$ and $D_{\beta\beta}$ respectively, then the damping matrix of the system can be written as $D = diag(0\, D_{\theta\theta}\ \ D_{\beta\beta}\ 0)$, where $diag(\bullet)$ denotes the diagonal matrix.

IV. SIMULATION

In this section, the simulation is used to verify the model built in the section above. The MATLAB software is used to solve the dynamical equation. The rotor type in the simulation is OA212-207, and the main parameters are shown in Table 1.

The initial values are as follows: the rotor shaft rotates with the speed of 36.63rad/s, the elastic deformation is 0, the initial acceleration can be derived by $\ddot{q}_0 = M_0^{-} Q_{F0}$, where M0 is the initial mass matrix, Q_{F0} is the initial generalized external force.

In the simulation, the viscous damping is used to simulate the air friction within Rotating surface of the rotor blade. The mass matrix, partial derivative of mass matrix on time, partial derivative of mass matrix on generalized coordinate, stiffness matrix, damping matrix are input to matlab, and the violation correction method is used to get the numerical solution of the dynamical equation. The Iterative step is 10^{-4}, and the iterative time is 2s.

TABLE I
MAIN PARAMETERS

Parameter	Value
Deadweight	M=1975kg
Radius of the rotor	l=5.5m
Number of the blades	n=4
Torsion	Fz=35000N/m
Moment of Inertia on blade	Iz=6726kg/m
Stiffness coefficient of torsion	$k_\beta = 18000 N/m$
Damping coefficients of the rotor shaft	$D_{\theta\theta}$=1600
Radius of the hub	r=0.5m

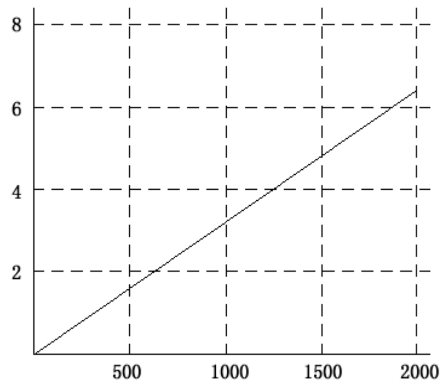

Fig. 2. The Curve of β

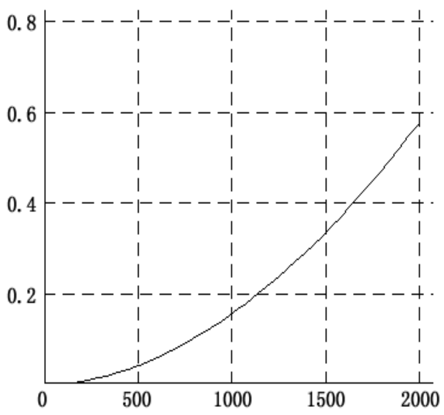

Fig. 3. The Curve of φ

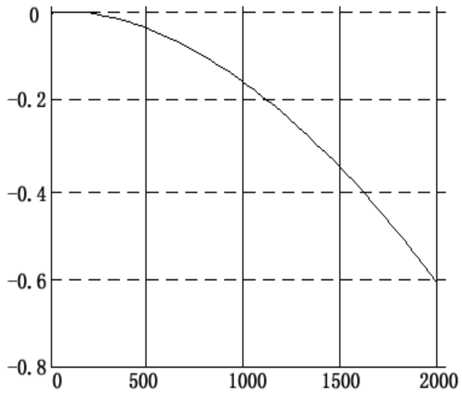

Fig. 4. The Curve of θ

Fig. 2 shows the angular displacement of the hub around the main axis. The angular displacement of the hub is a straight line, and the rotational speed descends slowly with small oscillation. Fig. 3 shows the angular displacement of the hub around dynamic axis and Fig. 4 shows the angular displacement of the impeller around dynamic axis. It can be seen that the two angular displacements are quadratic. At the end of time t = 2s, the angular displacement of the hub around the dynamic axis and the angular displacement of the impeller around dynamic axis are both agreement with the expected results, which indicates that the coordinate transformation matrix theory in this paper is correct.

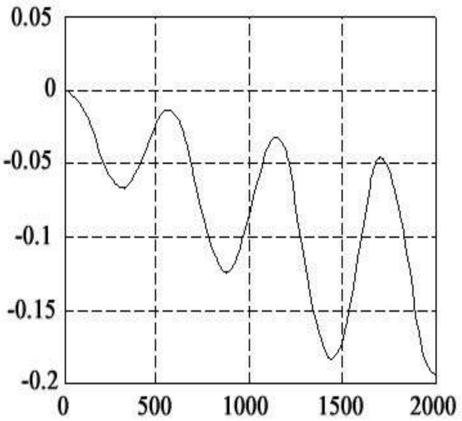

Fig. 5. Angular displacement with in the y-direction

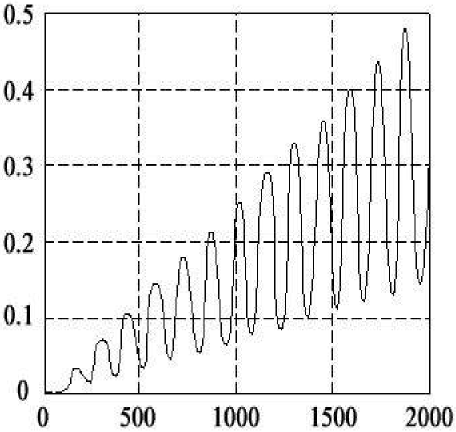

Fig. 6. Deformation with in y-direction

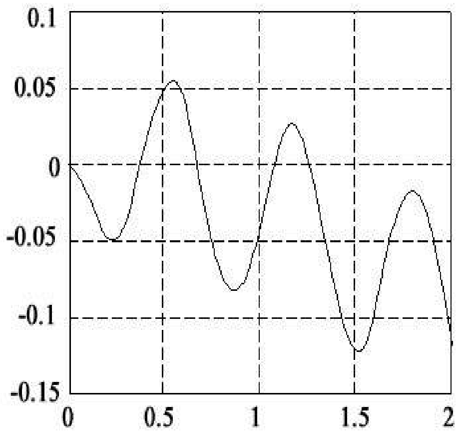

Fig. 7. Angular displacement with in the z-direction

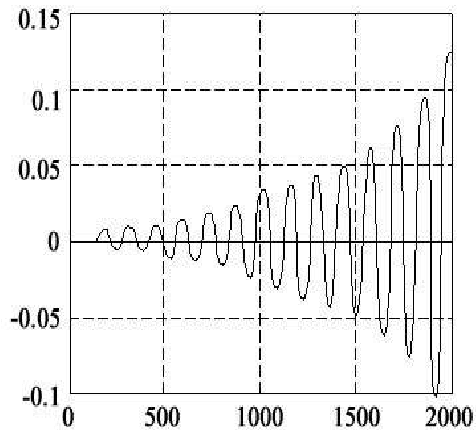

Fig.8. Deformation with in z-direction

Fig. 5 and Fig. 7 show the angular displacement of the blade end with in the y-direction and z-direction respectively, and Fig. 6 and Fig. 8 show the deformation of the blade end with in y-direction and z-direction respectively. From these figures, we can see that the a certain elastic deformation of the blade emerges due to its elasticity during the rotation of the rotor blade, and the elastic deformation can make the blade end deviates from a predetermined trajectory which will affect the positioning accuracy. Thus, the elastic deformation should be considered to make the rotational of the blade more accurate in practice. The vertical vibration cycle of the blade end in the plane which is perpendicular to the rotation plane is one seventeenth of that of the rotor shaft, and has lower frequency than that of the blade end in the rotation plane. These verify that the theoretical formula in this paper is correct.

V. CONCLUSION

The dynamical equation of the helicopter rotor blade system is built based on the finite element method and the Lagrange equation. Also, the mass matrix and its derivative on time are obtained based on the flexible multibody dynamics. In the simulation, the violation correction method is used to get the numerical solution of the dynamical equation. Simultaneously, the angular displacement of the hub around the main axis, the angular displacement of the blade end and elastic deformation in the y-direction and z-direction are got and analyzed to verify that the model in this paper is correct.

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